Numerical Analysis of Physics-Informed Neural Networks for Solving PDEs

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Abstract

This thesis presents an investigation into the theory, analysis, and enhancement of Physics-Informed Neural Networks (PINNs) for solving partial differential equations (PDEs). While PINNs have gained popularity for approximating PDE solutions using machine learning, their mathematical foundations, error control mechanisms, and integration with classical numerical methods remain underdeveloped. This work addresses these gaps by developing novel energy-norm and residual-based error estimates, thereby providing the first precise convergence characterizations of PINNs in Sobolev spaces. These bounds not only establish the well-posedness of PINN formulations but also enable their use in a posteriori error estimation and adaptive refinement schemes.

Building on this foundation, the thesis introduces a hybrid numerical framework that utilizes PINN residuals to identify high-error regions in Finite Difference Method (FDM) solutions to nonlinear PDEs, as a way to guide adaptive mesh refinement. The resulting FDM solvers are shown to achieve significantly improved accuracy with fewer degrees of freedom for Burgers' and the 2D incompressible Navier-Stokes equations. This constitutes a novel and practical use of PINNs not merely as solvers, but as refinement advisors for classical methods.

In addition, the thesis addresses the computational bottlenecks inherent in training large-scale PINNs by introducing a hierarchical matrix (H-matrix) compression strategy for the residual Jacobian and neural tangent kernel. This structured approximation significantly reduces memory usage and computational cost while preserving the spectral and convergence properties of the original network. Theoretical guarantees are provided for spectral stability, training error propagation, and convergence recovery under adaptive compression. These methods allow for energy-efficient PINN deployment on constrained hardware, enabling real-time PDE inference without compromising accuracy.

Beyond algorithmic improvements, the thesis develops a rigorous mathematical framework for analyzing PINNs as numerical solvers. Specifically, it establishes formal definitions and proofs for energy-norm convergence, variational stability, and operator consistency in the context of both linear and nonlinear PDEs. Using tools from functional analysis and variational theory, the work derives upper and lower bounds on solution error based on the empirical residual, and shows that under mild regularity assumptions, PINNs exhibit convergence properties analogous to Galerkin methods. A sequence of results; including coercivity-driven

error bounds, Ritz-type projection identities, and Γ-convergence of the discrete residual loss; demonstrate that residual minimization in PINNs leads to convergence in Sobolev and uniform norms. These results position PINNs as a basis-free, adaptive numerical scheme that inherits the core theoretical guarantees of classical methods, while remaining flexible and mesh-free. This synthesis of theory and practice offers a principled framework for reliable PINN deployment in scientific and engineering problems governed by PDEs.

Ekiwandiiko kino mu bufunze

Ekiwandiiko kino kinoonyereza ku ngeri moodulo ezemikutu ezikozesa amateeka gobutonde (Physics-Informed Neural Networks) mu nsengeka z'okubala ezesigamiziddwa ku bintu ebisukka ekimu (partial differential equations) gyezeeyisaamu, ate n'okulambulula ku mitendera emituufu egy'okusikozesaamu, nga zeesigika.

Moodulo zino eziddukkanyizibwa ku kompyuta, ziyiga engeri y'okulambikamu n'okuwa ebyokuddamu byensengeka (solutions to equations) nga ziyigirizibwa okusoma amateeka g'obutonde agali mu nsengeka ezo. Moodulo zino zikozesebwa okunoonya ebyokuddamu eri ensengeka enzibu ezikozesebwa okwoleka engeri ebintu gye bikyuka n'obudde wamu n'ekiseera oba gyebikyuuka n'ebintu ebirala. Okugeza engeri ebbugumu gye lisaasaana, engeri amayengo gye gatambulamu, oba engeri amazzi gye gatambulamu.

Ekitundu ekisooka eky'ekiwandiiko kino kiraga engeri moodulo zino gyeziyambako moodulo enkadde zi kaasangwawo, okutumbuula enkola y'azikaasangwawo. Mu kino tulambika obukakafu obusooka obutegeerekeka obulaga ddi n'engeri moodulo zino gye ziyambamu moodulo zikaasangwawo okwongera ku butuufu bwazo. Twoleka engeri okulemererwa kwa zikasaangwawo naddala ezeesigamiziddwa ku budaala oba giridi (grid) gye kudaabulurwa n'okukendezebwaamu nga tukoseza moodulo zino ez'emikutu. Tulaga nti moodulo ez'emikutu n'amateeka g'obutonde zisobola okulaga moodulo zikaasangwawo wezilemererwa. Olwo obudaala nebutereezebwa mu bitundu ebyo byokka. Kino kitegeeza nti okukozesa moodulo zino zombi kisingako mu butuufu wamu n'okukekkereza obudde n'ebikozesebwa.

Ekitundu eky'okubiri kissa essira ku nkozesa ennungamu eya zi moodulo zino mu budde obusaamusaamu. Okutendeka moodulo zino ez'emikutu n'amateeka g'obutonde ebiseera ebisinga kyetaagisa obudde bungi wamu n'ekitundu ku bwongo bwa kompyuta nga kiwereera ddala. Okusobola okugonjoola kino, ekiwandiiko kino kyanjula enkola y'okunyigiriza (Hierarchical matrices) ekendeeza ku bunene n'okuzitowa kwa moodulo z'emikutu. Kino kizisobozesa okutendekebwa mu bwangu, era nezikozesa obwongo bwa kompyuta obusaamusaamu n'ebiseera ebisaamusaamu.

Ekiwandiiko kino kifundikira n'okuwa omusingi omunywevu ogw'okukozesa moodulo z'emikutu, wamu n'okunnyonnyola lwaki zikola – mu ngeri gyezikolamu. Tulaga ebilina okutuukirizibwa moodulo zino okusobola okukozesebwa mu butuufu, nga ebyokuddamu ebizivaamu byesigika. Ku kino, tulambika amateeka n'obukwakkulizo obulina okukakasibwa nga ebiva mu moodulo zino tebineesigibwa. Nga ovude kwekyo, tulaga engeri y'okutegeeramu obugumu, n'obunywevu bwa zi moodulo zino okusinziira mu mateeka ge ssomo ly'okubala erinoonyereza ku masangazira g'okubala ne zi kompyuta.

Tumaliriza n'okulaga nti moodulo zino zeesigika era zibandaala n'ebyokuddamu ebyobuzaaliranwa era ebitukiridde (exact solutions) ebyalibadde bisuubirwa okuva mu nseengeka zebyo byezikozeeeko. Kino kiwa obwesige nti moodulo zino zigoberera amateeka ge ssomo ly'okubala erinonoonyereza ku masangazira g'okubala ne zi kompyuta. Ebivaamu ebirala bitulaga nti moodulo zino zituukana n'omutindo n'ezo i kaasangwawo.

Ebitundu bino byonsatule byongera okuwa abanoonyereza omusingi oumgumu era omwesimbu okukozesa moodulo zino okungonjoola, okunoonyereza n'okufuna eby'okuddamu mu ssaayansi ne'byobuyiiya